New class of Meromorphic multivalent functions by using derivative operator

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Abstract:

In this paper, we have introduced a new class $S^{\lambda,m}(\vartheta,\alpha,\mu)$ of meromorphic multivalent functions defined by Ruscheweyh derivative operator. We also obtained some geometric properties such as coefficient estimates, extreme points, convex set, distortion and covering theorem, δ-neighborhoods, partial sums and arithmetic mean. All results are sharp.

Keywords: Meromorphic Function, Multivalent Function, Derivative Operator, Distortion, Extreme Points, Arithmetic Mean etc.

1 Introduction

Let A_m be the class of functions of the form,

$f(z) = z^{-m} + \sum_{k=1}^{\infty} a_{k-m} z^{k-m}, \quad a_{k-m} \geq 0, m \in N$

which are analytic and meromorphic multivalent in the punctured unit disc $U^* = \{z \in C: 0 < |z| < 1\}$.

Consider the subclass T_m of the function of the form $f(z) = z^{-m} + \sum_{k=1}^{\infty} a_{k-m} z^{k-m}, \quad a_{k-m} \geq 0, m \in N$ (1)

The convolution of two functions, $f(z)$ is given by (1) and $g(z) = z^{-m} + \sum_{k=1}^{\infty} b_{k-m} z^{k-m}, \quad b_{k-m} \geq 0$ is defined by

$$(f * g)(z) = z^{-m} + \sum_{k=1}^{\infty} a_{k-m} b_{k-m} z^{k-m}, \quad a_{k-m} b_{k-m} \geq 0$$

We shall required Ruscheweyh derivative operator for the function belonging to the class T_m which is defined by the following convolution, $D^{\lambda,m} = \frac{z^{-m}}{(1-z)^{\lambda+m}} * f(z), \quad \lambda > -m, f \in T_m$ (2)

In terms of binomial coefficients (2) can be written as

$D^{\lambda,m} = z^{-m} + \sum_{k=1}^{\infty} \binom{\lambda+k}{k} a_{k-m} z^{k-m} \qquad \lambda > -m, f \in T_m$ (3)

The linear operator $D^{\lambda,1}$ was studied by Raina and Srivastava (2006). Also the operator $D^{\lambda,p}{}_*$ analogous to $D^{\lambda,m}$was studied by Goyal Prajapat (2006) and by W. G. Mustafa and Mouajeeb (2013).

A function $f \in T_m$ is meromorphic multivalent starlike function of order ρ, $0 \le \rho < m$ if $-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \qquad (0 \le \rho < m, z\epsilon U^*)$ (4)

A function $f \in T_m$ is meromorphic multivalent convex function of order ρ, $0 \le \rho < m$ if $-Re\left\{1 + \frac{zf''(z)}{f(z)}\right\} > \rho \qquad (0 \le \rho < m, z\epsilon U^*)$ (5)

Definition (01): Let $f \in T_m$ is given by (1). The class $S^{\lambda,m}(\vartheta, \alpha, \mu)$ is defined by

$$S^{\lambda,m}\left\{f \in T_m : \left|\frac{\vartheta\left(\left(D^{\lambda,m}f(z)\right)' - \frac{D^{\lambda,m}f(z)}{z}\right)}{\alpha\left(D^{\lambda,m}f(z)\right)' + (1-\vartheta)\frac{D^{\lambda,m}f(z)}{z}}\right| < \mu,\ 0 \le \vartheta < 1\ \ 0 \le \alpha < 1, 0 < \mu < 1, \lambda > -m, m \in N\right\}$$
(6)

2 COEFFICIENT INEQUALITY

Theorem (01): Let $f \in T_m$ the $f \in S^{\lambda,m}(\vartheta, \alpha, \mu)$ if and only if

$$\sum_{k=1}^{\infty}\binom{\lambda + k}{k}[\vartheta(k - m - 1) - \mu(\alpha(k - m) + 1 - \vartheta)]a_{k-m} \le \mu(1 - m\alpha - \vartheta) - \vartheta(m + 1)$$

$(0 \le \vartheta < 1\ \ 0 \le \alpha < 1, 0 < \mu < 1, \lambda > -m, m \in N)$ (7)

The result is sharp for the function $f(z) = z^{-m} + \sum_{k=1}^{\infty}\frac{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}z^{k-m}$.

Proof: Assume that the inequality (7) is hold true and let $|z| = 1$ then from (6) we have

$$\left|\vartheta\left(\left(D^{\lambda,m}f(z)\right)' - \frac{D^{\lambda,m}f(z)}{z}\right)\right| - \mu\left|\alpha\left(D^{\lambda,m}f(z)\right)' + (1-\vartheta)\frac{D^{\lambda,m}f(z)}{z}\right|$$

$$= \left|\vartheta\sum_{k=1}^{\infty}\binom{\lambda + k}{k}(k - m - 1)a_{k-m}z^{k-m} - (m + 1)\vartheta z^{-m-1}\right| - \mu\left|(1 - m\alpha - \vartheta)z^{-m-1} + \sum_{k=1}^{\infty}\binom{\lambda + k}{k}(\alpha(k - m) + 1 - \vartheta)a_{k-m}z^{k-m}\right|$$

$$\le \sum_{k=1}^{\infty}\binom{\lambda + k}{k}[\vartheta(k - m - 1) - \mu(\alpha(k - m) + 1 - \vartheta)]a_{k-m} - \mu(1 - m\alpha - \vartheta) + \vartheta(m + 1) \le 0.$$

Hence by maximum modulus principle, $f \in S^{\lambda,m}(\vartheta, \alpha, \mu)$.

Conversely, suppose that f defined by (1) is in the class $S^{\lambda,m}(\vartheta, \alpha, \mu)$.

Hence, $\left|\frac{\vartheta\left(\left(D^{\lambda,m}f(z)\right)'-\frac{D^{\lambda,m}f(z)}{z}\right)}{\alpha\left(D^{\lambda,m}f(z)\right)'+(1-\vartheta)\frac{D^{\lambda,m}f(z)}{z}}\right|$

$$=\left|\frac{-\vartheta(m+1)z^{-m}+\vartheta\sum_{k=1}^{\infty}(k-m-1)\binom{\lambda+k}{k}a_{k-m}z^{k-m-1}}{(1-m\alpha-\vartheta)z^{-m-1}+\sum_{k=1}^{\infty}(\alpha(k-m)+1-\vartheta)\binom{\lambda+k}{k}a_{k-m}z^{k-m-1}}\right|<\mu.$$

Since $Re(z)<|z|$ for all z we have,

$$Re\left\{\frac{\vartheta\sum_{k=1}^{\infty}(k-m-1)\binom{\lambda+k}{k}a_{k-m}z^{k-m}-\vartheta(m+1)z^{-m-1}}{(1-m\alpha-\vartheta)z^{-m-1}+\sum_{k=1}^{\infty}(\alpha(k-m)+1-\vartheta)\binom{\lambda+k}{k}a_{k-m}z^{k-m-1}}\right\} \tag{8}$$

Let $z\to 1^{-}$ through real values, so we can write (8) as,

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]a_{k-m}\le\mu(1-m\alpha-\vartheta)-\vartheta(m+1).$$

Finally sharpness follows if we take,

$$f(z)=z^{-m}+\sum_{k=1}^{\infty}\frac{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}z^{k-m},\qquad k\ge 1.$$

Corollary (01):

Let $f\in S^{\lambda,m}(\vartheta,\alpha,\mu)$ then $a_{k-m}\le\frac{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}$ where

$0\le\vartheta<1$ $0\le\alpha<1, 0<\mu<1, \lambda>-m, m\in N$.

3 CONVEX SET

Theorem (02): Let the functions $f(z)=z^{-m}+\sum_{k=1}^{\infty}a_{k-m}z^{k-m}$, $a_{k-m}\ge 0$

$g(z)=z^{-m}+\sum_{k=1}^{\infty}b_{k-m}z^{k-m}$, $b_{k-m}\ge 0$ be in the class $S^{\lambda,m}(\vartheta,\alpha,\mu)$. Then for $0\le l\le 1$, the function $d(z)=(1-l)f(z)+l\,g(z)=z^{-m}+\sum_{k=1}^{\infty}c_{k-m}z^{k-m}$ (9)

Where $c_{k-m}=(1-l)a_{k-m}+lb_{k-m}\ge 0$ is also in the class $S^{\lambda,m}(\vartheta,\alpha,\mu)$.

Proof: Suppose that each of the functions f and g is in the class $S^{\lambda,m}(\vartheta,\alpha,\mu)$. Then making use of theorem (01) we see that,

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]c_{k-m}$$

$$=(1-l)\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]a_{k-m}$$

$$+l\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]a_{k-m}$$

$$\leq (1-l)[\mu(1-m\alpha-\vartheta)-\vartheta(m+1)] + l[\mu(1-m\alpha-\vartheta)-\vartheta(m+1)]$$

$\leq [\mu(1-m\alpha-\vartheta)-\vartheta(m+1)]$, which completes the proof.

4 EXTREME POINTS

Theorem (03): Let $f_{-m} = z^{-m}$, and

$$f_{k-m}(z) = z^{-m} + \frac{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]} z^{k-m} \tag{10}$$

For $k = 1,2,\ldots$ Then $f \in S^{\lambda,m}(\vartheta,\alpha,\mu)$ if and only if it can be expressed in the form,

$f(z) = \sum_{k=0}^{\infty} d_{k-m} f_{k-m}(z)$, where $d_{k-m} \geq 0$ *and* $\sum_{k=0}^{\infty} d_{k-m} = 1$.

Proof: Suppose that $f(z) = \sum_{k=0}^{\infty} d_{k-m} f_{k-m}(z)$ where $d_{k-m} \geq 0$ *and* $\sum_{k=0}^{\infty} d_{k-m} = 1$.

Then

$f(z) = d_{-m} f_{-m}(z) + \sum_{k=1}^{\infty} d_{k-m} f_{k-m}(z)$.

$$= d_{-m} z^{-m} + \sum_{k=1}^{\infty} d_{k-m} \left(z^{-m} + \frac{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]} z^{k-m} \right)$$

$$= z^{-m} + \sum_{k=1}^{\infty} \frac{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]} z^{k-m}$$

$= z^{-m} + \sum_{k=1}^{\infty} P_{k-m} z^{k-m}$ where $P_{k-m} = \frac{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}$

By theorem (01), we have $f \in S^{\lambda,m}(\vartheta,\alpha,\mu)$ if and only if $\sum_{k=1}^{\infty} \frac{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)} P_{k-m} \leq 1$,

For $f(z) = z^{-m} + \sum_{k=1}^{\infty} P_{k-m} z^{k-m}$

Hence $\sum_{k=1}^{\infty} \frac{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)} \times d_{k-m} \frac{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}$

$$= \sum_{k=1}^{\infty} d_{k-m} = 1 - d_{-m} \leq 1$$

The proof is complete.

Conversely, assume that $f \in S^{\lambda,m}(\vartheta,\alpha,\mu)$. Then we can show that f can be written in the form

$f(z) = \sum_{k=0}^{\infty} d_{k-m} f_{k-m}(z)$.

Now $f \in S^{\lambda,m}(\vartheta,\alpha,\mu)$

Therefore from theorem (01)

$$a_{k-m} \le \frac{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}$$

Setting

$$d_{k-m} = \frac{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)} a_{k-m} \qquad k = 1,2,\ldots$$

And

$$d_{-m} = 1-\sum_{k=1}^{\infty} d_{k-m}$$

Then $f(z) = z^{-m} + \sum_{k=1}^{\infty} a_{k-m} z^{k-m}$

$$f(z) = z^{-m} + \sum_{k=1}^{\infty} \frac{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]} d_{k-m}$$

$$= z^{-m} + \sum_{k=1}^{\infty} (f_{k-m}-z^{-m}) d_{k-m}$$

$$= z^{-m}\left(1-\sum_{k=1}^{\infty} d_{k-m}\right) + \sum_{k=0}^{\infty} d_{k-m} f_{k-m}$$

$$= z^{-m} d_{-m} + \sum_{k=1}^{\infty} d_{k-m} f_{k-m}$$

$$= \sum_{k=0}^{\infty} d_{k-m} f_{k-m}(z)$$

5 DISTORTION AND COVERING THEOREM

Theorem (04): If the function $f \in S^{\lambda,m}(\vartheta,\alpha,\mu)$ then for $0 < |z| < 1$

$$\frac{1}{|z|^m} - \frac{\vartheta(m+1)-\mu(1-m\alpha-\vartheta)}{\binom{\lambda+1}{1}[\vartheta m+\mu(\alpha(1-m)+1-\vartheta)]}|z|^{1-m} \le |f(z)| \le$$

$$\frac{1}{|z|^m} + \frac{\vartheta(m+1)-\mu(1-m\alpha-\vartheta)}{\binom{\lambda+1}{1}[\vartheta m+\mu(\alpha(1-m)+1-\vartheta)]}|z|^{1-m} \qquad (11)$$

The result is sharp and obtained for $f(z) = \frac{1}{|z|^m} + \frac{\vartheta(m+1)-\mu(1-m\alpha-\vartheta)}{\binom{\lambda+1}{1}[\vartheta m+\mu(\alpha(1-m)+1-\vartheta)]}|z|^{1-m}$.

Proof: Let $f \in S^{\lambda,m}(\vartheta,\alpha,\mu)$ then

$$|f(z)| = \left| z^{-m} + \sum_{k=1}^{\infty} a_{k-m} z^{k-m} \right|$$

$$\le \frac{1}{|z|^m} + \sum_{k=1}^{\infty} a_{k-m}|z|^{k-m}$$

$$\leq \frac{1}{|z|^m} + |z|^{1-m} \sum_{k=1}^{\infty} a_{k-m}$$

Therefore by theorem (01),

$$a_{k-m} \leq \frac{\vartheta(m+1) - \mu(1 - m\alpha - \vartheta)}{\binom{\lambda+1}{1}[\vartheta m + \mu(\alpha(1-m) + 1 - \vartheta)]}$$

Therefore

$$|f(z)| \leq \frac{1}{|z|^m} + \frac{\vartheta(m+1) - \mu(1 - m\alpha - \vartheta)}{\binom{\lambda+1}{1}[\vartheta m + \mu(\alpha(1-m) + 1 - \vartheta)]}|z|^{1-m}$$

Similarly, we have

$$|f(z)| \geq \frac{1}{|z|^m} - \frac{\vartheta(m+1) - \mu(1 - m\alpha - \vartheta)}{\binom{\lambda+1}{1}[\vartheta m + \mu(\alpha(1-m) + 1 - \vartheta)]}|z|^{1-m}.$$

Theorem (05): If the function $f \in S^{\lambda,m}(\vartheta, \alpha, \mu)$ then for $0 < |z| < 1$

$$\frac{m}{|z|^{m+1}} - \frac{[\vartheta(m+1) - \mu(1 - m\alpha - \vartheta)](1-m)}{\binom{\lambda+1}{1}[\vartheta m + \mu(\alpha(1-m) + 1 - \vartheta)]}|z|^{-m} \leq |f'(z)| \leq$$

$$\frac{m}{|z|^{m+1}} + \frac{[\vartheta(m+1) - \mu(1 - m\alpha - \vartheta)](1-m)}{\binom{\lambda+1}{1}[\vartheta m + \mu(\alpha(1-m) + 1 - \vartheta)]}|z|^{-m} \qquad (12)$$

The result is sharp and obtained for $f(z) = \frac{m}{|z|^{m+1}} + \frac{[\vartheta(m+1) - \mu(1 - m\alpha - \vartheta)](1-m)}{\binom{\lambda+1}{1}[\vartheta m + \mu(\alpha(1-m) + 1 - \vartheta)]}|z|^{-m}$.

Proof: Let $f \in S^{\lambda,m}(\vartheta, \alpha, \mu)$ then

$$|f(z)| = \left| z^{-m} + \sum_{k=1}^{\infty} a_{k-m} z^{k-m} \right|$$

$$|f'(z)| = \left| -mz^{-m-1} + \sum_{k=1}^{\infty} (k-m) a_{k-m} z^{k-m-1} \right|$$

$$\leq \frac{m}{|z|^{m+1}} + \sum_{k=1}^{\infty} (k-m) a_{k-m} |z|^{k-m-1}$$

$$\leq \frac{m}{|z|^{m+1}} + |z|^{-m} \sum_{k=1}^{\infty} (1-m) a_{k-m}$$

By theorem (01), we have

$$|f'(z)| \le \frac{m}{|z|^{m+1}} + \frac{[\vartheta(m+1) - \mu(1 - m\alpha - \vartheta)](1-m)}{\binom{\lambda+1}{1}[\vartheta m + \mu(\alpha(1-m) + 1 - \vartheta)]}|z|^{-m}$$

Similarly, we have

$$|f'(z)| \ge \frac{m}{|z|^{m+1}} - \frac{[\vartheta(m+1) - \mu(1 - m\alpha - \vartheta)](1-m)}{\binom{\lambda+1}{1}[\vartheta m + \mu(\alpha(1-m) + 1 - \vartheta)]}|z|^{-m}$$

which complete the proof.

6 ARITHMATIC MEAN

Theorem (06): Let $f_1(z), f_2(z) \dots f_n(z)$ defined by

$$f_i(z) = z^{-m} + \sum_{k=1}^{\infty} a_{k-m,i} z^{k-m} \qquad (a_{k-m,i} \ge 0, i = 1,2, \dots n, k \ge 1) \tag{13}$$

be in the class $S^{\lambda,m}(\vartheta, \alpha, \mu)$. Then the arithmetic mean of $f_i(z)$ $(i = 1,2, \dots n)$ is defined by

$$h(z) = \frac{1}{n}\sum_{i=1}^{n} f_i(z) \tag{14}$$

is also in the class $S^{\lambda,m}(\vartheta, \alpha, \mu)$.

Proof: By (13) and (14) we can write

$$h(z) = \frac{1}{n}\sum_{i=1}^{n}\left(z^{-m} + \sum_{k=1}^{\infty} a_{k-m,i} z^{k-m}\right)$$

$$= z^{-m} + \sum_{k=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_{k-m,i}\right) z^{k-m}$$

Since $f_i \in S^{\lambda,m}(\vartheta, \alpha, \mu)$ for every $(i = 1,2, \dots n)$ so by theorem (01),

We prove that

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[\vartheta(k-m-1) - \mu(\alpha(k-m) + 1 - \vartheta)]\left(\frac{1}{n}\sum_{i=1}^{n} a_{k-m,i}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[\vartheta(k-m-1) - \mu(\alpha(k-m) + 1 - \vartheta)]a_{k-m,i}\right)$$

$$\le \frac{1}{n}\sum_{i=1}^{n}\mu(1 - m\alpha - \vartheta) - \vartheta(m+1)$$

$$= \mu(1 - m\alpha - \vartheta) - \vartheta(m+1)$$

Therefore $h(z) \in S^{\lambda,m}(\vartheta,\alpha,\mu)$.

7 δ NEIGHBORHOODS

Definition (02): Let $(0 \leq \vartheta < 1\ 0 \leq \alpha < 1, 0 < \mu < 1, \lambda > -m, m \in N)$ and $\delta \geq 0$ we define δ neighborhood of function $f \in T_m$ and denote $N_\delta(f)$ such that

$$N_\delta(f) = \left\{g \in T_m : g(z) = z^{-m} + \sum_{k=1}^{\infty} b_{k-m} z^{-m}\ and \sum_{k=1}^{\infty} \frac{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)} |a_k - b_k| \leq \delta\right\}. \quad (15)$$

Theorem (07): Let function $f \in T_m$ be in the class $S^{\lambda,m}(\vartheta,\alpha,\mu)$, for every complex number β with $|\beta| < \delta, \delta \geq 0$.

Let $\frac{f(z)+\beta z^{-m}}{1+\beta} \epsilon\, S^{\lambda,m}(\vartheta,\alpha,\mu)$ then $N_\delta(f) \subset S^{\lambda,m}(\vartheta,\alpha,\mu)$, $\quad \delta \geq 0$.

Proof: Since $f(z) \in S^{\lambda,m}(\vartheta,\alpha,\mu)$, f satisfies (7) and we can write for $n \in C, |n| = 1$, that

$$\left[\frac{\vartheta\left(\left(D^{\lambda,m} f(z)\right)' - \frac{D^{\lambda,m} f(z)}{z}\right)}{\alpha\left(D^{\lambda,m} f(z)\right)' + (1-\vartheta)\frac{D^{\lambda,m} f(z)}{z}}\right] \neq n \quad (16)$$

Equivalently, we must have $\frac{(f*Q)(z)}{z^{-m}} \neq 0$, $\quad z \in U^*$ (17)

Where $Q(z) = z^{-m} + \sum_{k=1}^{\infty} e_{k-m} z^{k-m}$

Such that $e_{k-m} = \frac{n\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}$

Satisfying $|e_{k-m}| \leq \frac{n\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}$ and $k \geq 1, m \in N$

Since $\frac{f(z)+\beta z^{-m}}{1+\beta} \epsilon S^{\lambda,m}(\vartheta,\alpha,\mu)$

By (17) $\frac{1}{z^{-m}}\left(\frac{f(z)+\beta z^{-m}}{1+\beta} * Q(z)\right) \neq 0$ (18)

Now we assume that, $\left|\frac{(f*Q)z}{z^{-m}}\right| < \delta$ then by (18), we have

$$\left|\frac{1}{1+\beta}\frac{(f*Q)z}{z^{-m}} + \frac{\beta}{1+\beta}\right| \geq \frac{|\beta|}{|1+\beta|} - \frac{1}{|1+\beta|}\left|\frac{(f*Q)z}{z^{-m}}\right| > \frac{|\beta|-\delta}{|1+\beta|} \geq 0$$

This is the contradiction as $|\beta| < \delta$.

Therefore $\left|\frac{(f*Q)z}{z^{-m}}\right| \geq \delta$

Letting $g(z) = z^{-m} + \sum_{k=1}^{\infty} b_{k-m} z^{k-m} \in N_{\delta}(f)$

Then $\delta - \left|\frac{(g*Q)z}{z^{-m}}\right| \leq \left|\frac{(f-g)*Q(z)}{z^{-m}}\right|$

$$\leq \left|\sum_{k=1}^{\infty} (a_{k-m} - b_{k-m}) e_{k-m} z^{k-m}\right|$$

$$\leq \sum_{k=1}^{\infty} |a_{k-m} - b_{k-m}||e_{k-m}||z|^{k-m}$$

$$< |z|^{k-m} \sum_{k=1}^{\infty} \left[\frac{\binom{\lambda+k}{k}[\vartheta(k-m-1) - \mu(\alpha(k-m)+1-\vartheta)]}{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}\right] |a_{k-m} - b_{k-m}| \qquad \leq \delta$$

Therefore, $\left|\frac{(g*Q)z}{z^{-m}}\right| \neq 0$ we get $g(z) \in S^{\lambda,m}(\vartheta, \alpha, \mu)$

So $N_{\delta}(f) \subset S^{\lambda,m}(\vartheta, \alpha, \mu)$, $\qquad \delta \geq 0$.

8 PARTIAL SUM

Theorem (08): Let f(z) is defined by (1) and the partial sum$S_1(z)$ and $S_q(z)$ be defined by $S_1(z) = z^{-m}$and $S_q(z) = z^{-m} + \sum_{k=1}^{q-1} a_{k-m} z^{k-m} \qquad (q > 1)$.

Also suppose that, $\sum_{k=1}^{\infty} c_{k-m} a_{k-m} \leq 1$

Where $c_{k-m} = \frac{\binom{\lambda+k}{k}[\vartheta(k-m-1)-\mu(\alpha(k-m)+1-\vartheta)]}{\mu(1-m\alpha-\vartheta)-\vartheta(m+1)}$ (19)

Then we have $Re\left\{\frac{f(z)}{S_q(z)}\right\} > 1 - \frac{1}{c_q}$ (20)

$Re\left\{\frac{f(z)}{S_q(z)}\right\} > 1 - \frac{c_q}{1+c_q}$, $\qquad (z \in U^*, q > 1)$ (21)

Each of the bounds in (19) and (20) is the best possible for $k \in N$.

Proof: For the coefficients c_{k-m} given by (19), it is not difficult to verify

$$c_{k-m+1} > c_{k-m} > 1, \quad k = 1,2 \dots$$

Therefore by using the hypothesis (19) we have

$\sum_{k=1}^{q-1} a_{k-m} + c_q \sum_{k=q}^{\infty} a_{k-m} \leq \sum_{k=1}^{\infty} c_{k-m} a_{k-m} \leq 1$ (22)

By setting, $G_1(z) = c_q\left(\frac{f(z)}{S_q(z)} - \left(1 - \frac{1}{c_q}\right)\right)$

$$= \frac{f(z)}{S_q(z)}c_q - c_q + 1$$

$$= \frac{c_q\big(f(z) - S_q(z)\big)}{S_q(z)} + 1$$

$$= \frac{c_q \sum_{k=q}^{\infty} a_{k-m} z^{k-m}}{z^{-m} + \sum_{k=1}^{q-1} a_{k-m} z^{k-m}} + 1$$

$$= \frac{c_q \sum_{k=q}^{\infty} a_{k-m} z^{k}}{1 + \sum_{k=1}^{q-1} a_{k-m} z^{k}} + 1$$

Applying (22) we find that

$$\left|\frac{G_1(z) - 1}{G_1(z) + 1}\right| = \left|\frac{c_q \sum_{k=q}^{\infty} a_{k-m} z^{k}}{c_q \sum_{k=q}^{\infty} a_{k-m} z^{k} + 2 + 2\sum_{k=1}^{q-1} a_{k-m} z^{k}}\right|$$

$$\leq \frac{c_q \sum_{k=q}^{\infty} a_{k-m}}{2 - 2\sum_{k=1}^{q-1} a_{k-m} - c_q \sum_{k=q}^{\infty} a_{k-m}} \leq 1$$

This proof (20). Therefore $Re(G_1(z)) > 0$ and we obtain

$$Re\left\{\frac{f(z)}{S_q(z)}\right\} > 1 - \frac{1}{c_q}.$$

Now, in this manner we can prove that the assertion (21) by Setting

$$G_2(z) = (1 + c_q)\left(\frac{S_q(z)}{f(z)} - \frac{c_q}{1 + c_q}\right)$$

This completes the proof.

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